FRICTIONAL RESISTANCE OF SPHERES ROTATING IN POWER-LAW NON-NEWTONIAN FLUIDS

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Frictional resistance coefficients have been calculated for the rotation of spheres in non-Newtonian power-law fluids under laminar boundary-layer conditions using the approximate method of integral momentum balances. The values obtained agree satisfactorily with the available experimental data and the published theoretical solution.

In a previous paper[1] the possibilities have been analyzed to use the method of integral momentum balances for the estimation of the frictional resistance of flat disks rotating under laminar boundary-layer flow conditions in an otherwise quiescent power-law fluid. Introducing suitably postulated families of velocity profiles (expressions for the azimuthal and radial components of the velocity vector within the boundary layer) we were able to achieve a fair coincidence between the torque values, calculated by the approximate method, and the values resulting from the exact solution. There was also found a good agreement of the results of the approximate method with the frictional resistance measured experimentally for the rotation of disks of finite dimensions in real pseudoplastic fluids.

The aim of this paper is to verify the reliability of the results which would be obtained if the screened-out profiles from ref.[1] are used in the analogous case – the laminar boundary-layer flow over spheres rotating in pseudoplastic power-law fluids.

THEORETICAL

We use a spherical reference frame r, θ, φ, where r is measured radially outwards from the centre of a sphere with the radius R, and θ from the vertical axis of steady rotation with angular velocity ω. The components of the velocity vector in the azimuthal (φ) and meridional (θ) directions are u and v. Applying the commonly used method, the integration of the boundary-layer equations across the boundary layer of thickness δ, taking advance of the continuity equation, using the relevant boundary conditions and introducing the normal coordiate, $z = r - R$, we obtain (see *e.g.*, ref.[2]) the following momentum equations:

$$\frac{d}{d\theta}\int_0^\delta uv\,dz + 2\int_0^\delta uv \cot g\,\theta\,dz = -\frac{R}{\varrho}(\tau_{\varphi z})_0 \qquad (1)$$

$$\frac{d}{d\theta}\int_0^\delta v^2\,dz + \int_0^\delta (v^2 - u^2)\cot g\,\theta\,dz = -\frac{R}{\varrho}(\tau_{\theta z})_0\,. \qquad (2)$$

In the given form, with shearing stresses at the wall, they are applicable to any inelastic non-Newtonian fluid with density ϱ. For power-law (Ostwald–de Waele) fluids, described by the flow index n and the consistency coefficient K, it holds:

$$(\tau_{\varphi z})_0 = K\left|\left(\frac{\partial u}{\partial z}\right)_0^2 + \left(\frac{\partial v}{\partial z}\right)_0^2\right|^{(n-1)/2}\left(\frac{\partial u}{\partial z}\right)_0 \qquad (3)$$

$$(\tau_{\theta z})_0 = K\left|\left(\frac{\partial u}{\partial z}\right)_0^2 + \left(\frac{\partial v}{\partial z}\right)_0^2\right|^{(n-1)/2}\left(\frac{\partial v}{\partial z}\right)_0\,. \qquad (4)$$

In analogy with the solution for flat disks[1], we shall use the following simple expressions (profiles) for the velocity distributions of the azimuthal and meridional components of the velocity vector within the bounary layer:

$$\frac{u}{a\omega} \equiv \mathscr{F}(Z) = (1 - Z)^l \qquad (5)$$

$$\frac{v}{\alpha a\omega \cos\theta} \equiv \mathscr{G}(Z) = [Z - mZ^m + (m-1)\,Z^{m+1}] \qquad (6)$$

with $a = R\sin\theta$ for the distance of the sphere surface from the axis rotation. In Eqs (5), (6), $Z(\theta) = z/\delta(\theta)$ is the normalized coordinate within the boundary layer and $\alpha(\theta)$ another function to be determined by solving the system (1) to (6). The exponents l and m in the velocity profiles (5), (6) can be functions of the flow index n, expressing thus the dependence of the exact velocity distributions on this parameter. Reasons for the selection of velocity profiles given just by Eqs (5), (6) can be found in ref.[1].

With the abbreviations

$$k_1 \equiv \int_0^1 \mathscr{G}^2\,dZ\,,\quad k_2 \equiv \int_0^1 \mathscr{F}^2\,dZ\,,\quad k_3 \equiv \int_0^1 \mathscr{F}\mathscr{G}\,dZ \qquad (7)$$

and the dimensionless (local) thickness of the boundary layer

$$\zeta_{0,a}(\theta) \equiv \delta \left[\frac{\omega^{2-n}(R \sin \theta)^{1-n}}{K/\varrho} \right]^{1/(n+1)} \tag{8}$$

it is possible to obtain from (*1*) to (*8*) (after somewhat lengthy, but obvious manipulations) the following system of two simultaneous first-order ordinary differential equations (ODE) in $\zeta_{0,a}$ and α:

$$\frac{d\zeta_{0,a}^{n+1}}{d\theta} = \frac{(n+1)(l^2 + \alpha^2 \cos^2 \theta)^{(n-1)/2}}{\alpha \cos \theta \sin \theta} \left(\frac{2l}{k_3} + \frac{1}{k_1} \right) - \\ - \left[2(3n+2) + \frac{k_2(n+1)}{\alpha^2 k_1 \cos^2 \theta} \right] \zeta_{0,a}^{n+1} \operatorname{cotg} \theta \tag{9}$$

$$\frac{d\alpha}{d\theta} = \alpha \left[1 + \frac{k_2}{\alpha^2 k_1 \cos^2 \theta} \right] \operatorname{cotg} \theta + \alpha \operatorname{tg} \theta - \frac{(l^2 + \alpha^2 \cos^2 \theta)^{(n-1)/2}}{\cos \theta \sin \theta \zeta_{0,a}^{n+1}} \left(\frac{l}{k_3} + \frac{1}{k_1} \right). \tag{10}$$

At small values of θ, *i.e.* in the vicinity of the poles of the sphere, Eqs (*9*), (*10*) reduce to a system of two nonlinear algebraic equations which are valid for the case of disk rotation in power-law fluids[1]. Their solution are the constant quantities ζ_D and α_D

$$\alpha_D = + \sqrt{\left(\frac{k_2(n+1)\, l}{k_3(5n+3) + l k_1(4n+2)} \right)} \tag{11}$$

$$\zeta_D = \left[\frac{(n+1)\, l(\alpha^2 + l^2)^{(n-1)/2}}{(5n+3)\, k_3 \alpha} \right]^{1/(n+1)} \tag{12}$$

which can be used as starting values in the numerical integration of the ODE-system (*9*), (*10*), the primary results of which are the dependencies of $\zeta_{0,a}$ and α over the interval $\theta \in (0, \pi/2)$. Now, it is already possible to determine the torque M_k, required to maintain a steady rotation of the sphere in the regime of boundary-layer flow in an infinite sea of an otherwise quiescent power-law fluid:

$$M_k = 4\pi R^{3+n} K l \omega^n \int_0^{\pi/2} (l^2 + \alpha^2 \cos^2 \theta)^{(n-1)/2} \frac{\sin^{2+n}\theta}{\delta^n} \, d\theta \,. \tag{13}$$

This relation can be rewritten into a dimensionless form, *i.e.* into the relation between the torque coefficient, c_M, and the generalized Reynolds number for rotational flows of power-law fluids, $\mathrm{Re}_{\mathrm{OW}}$:

$$c_{\mathrm{M}} = \frac{k(n)}{\mathrm{Re}_{\mathrm{OW}}^{1/(1+n)}}, \tag{14}$$

where

$$c_{\mathrm{M}} = \frac{2M_{\mathrm{k}}}{\varrho \omega^2 R^5} \tag{15}$$

$$\mathrm{Re}_{\mathrm{OW}} = \frac{R^2 \omega^{2-n}}{K/\varrho}. \tag{16}$$

The (approximate) frictional resistance coefficient k for sphere rotation is given as

$$k(n) = 8\pi l \int_0^{\pi/2} (l^2 + \alpha^2 \cos^2 \theta)^{(n-1)/2} \frac{\sin^{2(2n+1)/(n+1)}\theta}{\zeta_{0,\mathrm{a}}^n} \, \mathrm{d}\theta. \tag{17}$$

The assesment of the accuracy of the approximate momentum integral method can be best accomplished by comparison of the $k(n)$-values calculated according to (17) with their counterparts, resulting from the numerical solution of the problem considered[3,4] or with the results of experimental measurement of the frictional resistance of spheres rotating in pseudoplastic fluids[5].

RESULTS AND DISCUSSION

From an extensive series of calculations according to the above-mentioned relations only two sets of the approximate $k(n)$-values will be compared with their theoretical or experimental counterparts. Both these sets are based on the velocity profiles (5), (6); they differ one from another only in the dependencies of their exponents on the flow index which have been postulated as follows:

$$\text{Set } A: \quad l = 1 + 1/n \qquad m = 1 + 1/n \tag{18}$$

$$\text{Set } B: \quad l = 2/\sqrt{n} \qquad m = (5 - n)/2. \tag{19}$$

The set A corresponds to the velocity profiles, proposed by Tomita[6] in the first application of the momentum-integral method in the field of rotational boundary-layer flows of non-Newtonian (power-law) fluids. Set B turned out to be – at least from the viewpoint of the prediction of the dynamic responses – the "optimal" one in the previously mentioned detailed analysis of the problem of rotation of flat disks[1].

From the values given in Table I it follows, that the accuracy of the approximate computation of the frictional resistance coefficients for spheres rotating in the laminar boundary-layer flow regime in pseudoplastic fluids is sufficient for practical purposes. In analogy with disk-shaped spindles, the "best" estimates of the rotational resistance coefficients are obtained using velocity profiles of the type (*5*), (*6*) with the exponents l and m given by the rules (*19*), *i.e.* for the set B.

The value of the approximate frictional resistance coefficient $k(1) = 6{\cdot}75$ for Newtonian fluids ($n = 1$, *i.e.* $l = m = 2$, that means for $\mathscr{F} = (1 - Z)^2$ and $\mathscr{G} = Z(1 - Z)^2$) agrees surprisingly well with the theoretical solution due to Banks[3], $k(1) = 6{\cdot}54$, much better than with the approximate value $k(1) = 5{\cdot}95$ which was calculated by Sawatzki[2] additionally on the basis of the intermediate results of Howarth[7]. This author attacked the problem of boundary-layer flow over spheres rotating in Newtonian fluids by generalizing the approach of Kármán[8], *i.e.* by using polynoms of third and fourth order for the pertinent velocity profiles. These profiles satisfy more boundary conditions on the surface of the sphere than those used in this work. Therefore Howarth obtained a more complicated computational scheme which, however, did not lead in the end effect to better results in comparison with our simpler relations. Details of this aspect have been discussed in our previous work on disk rotation[1].

TABLE I

Comparison of the values of the frictional resistance coefficients k for different values of the flow index n

n	Theory (see refs[3,4])	Experiment (see ref.[5])	Momentum-integral method	
			Set A	Set B
1·0	6·54	6·73 ± 0·11	6·75 (+3·2)[b]	6·75 (+3·2)
0·8	7·62	—	7·81 (+3·1)	7·75 (+1·7)
0·5	10·08	10·2 ± 0·17	10·04 (−0·4)	9·92 (−1·6)
0·24	13·59[a]	13·5 ± 0·22	12·25 (−9·9)	13·12 (−3·5)

[a] Extrapolated from the empirical formula (*5*) in ref.[5]. [b] Values in brackets are the relative deviations in k between the approximate and the theoretical solution (in %).

CONCLUSION

This study shows that appropriate postulation of the velocity profiles in the method of integral momentum balances makes it possible to obtain easily (solving a system of two simultaneous first-order ODE as an initial value problem) reliable estimates of the frictional resistance coefficients for the rotation of spherical spindles in power--law non-Newtonian fluids. The exact method to find this practically interesting quantities requires a complicated numerical procedure (solution of multidimensional systems of highly nonlinear ODE as a boundary value problem, see ref.[4]).

On the basis of the results obtained with the approximate method for disk-shaped and spherical spindles it can be expected that an analogous approach could provide valuable informations on the dynamic responses (torque, frictional resistance) also for the case of rotational flows over other axisymmetrical bodies in pseudoplastic fluids in the laminar boundary-layer regime.

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